Kuwait University | Math. 101 | January 8th, 1994
Mathematics Dept. | Final Examination | Duration: 2 hours

Questions 1 to 6 are weighted equally at 14 points each, question 7 is worth 16 points. Calculators are not allowed.

1. Evaluate the limit, if it exists,

   (a) $\lim_{x\to 0} \dfrac{\sin^2(3x)}{x^3+x^2}$, (b) $\lim_{x\to +\infty} (\sqrt{x^2+2x} - x)$

2. (a) Let y be a function of x defined implicitly by $xy(xy+x-y)=1$. Find the equation of the normal line to the graph of y at $x=1$.

   (b) If $y = 3x^2 - 2\sqrt{x+1}$ and $x = t^3+t^2+1$, then find $\dfrac{dy}{dt}$ at $t=1$.

3. A wire $17cm$ long is cut into two pieces. One piece is bent into the shape of a square and the other into the shape of a rectangle whose length is twice its width. Where should the wire be cut so that the sum of the areas of the square and the rectangle is minimum.

4. (a) State Rolle's theorem. Let

$$f(x) = x^2 + 2\sqrt{9-x^2}\,, \quad -3 \le x \le 3.$$

   Find $c \in (-3,3)$ that satisfies the conclusion of Rolle's theorem for f.

   (b) Verify the trigonometric identity, $2\sin^2(2x) + \cos 4x = 1$ for all $x \in R$.

5. Evaluate the integrals: (a) $\displaystyle\int_0^1 \frac{x^2}{(7x^3+1)^{\frac{1}{3}}}\,dx$ (b) $\displaystyle\int (\sin 2x + \cos 2x)^2\,dx$.

6. (a) Let $f(x) = 2 + 3\sqrt{x}$, $1 \le x \le 4$. Find $z \in (1,4)$ that satisfies the conclusion of the Mean Value Theorem for definite integrals for the function f.

   (b) Show that the function $f(x) = \displaystyle\int_0^x \sqrt{t^{\frac{2}{3}} + \cos^4 t}\,dt$ is increasing on $[0, \frac{\pi}{2}]$.

7. Let $f(x) = 2 - \dfrac{4}{x} + \dfrac{6}{x^2}$,

   (a) Find the horizontal and the vertical asymptotes for the graph of f (if any).

   (b) Find the intervals where f is increasing and the intervals where f is decreasing.

   (c) Find the local extrema of f.

   (d) Find the intervals where the graph of f is concave upward, and the intervals where the graph of f is concave downward. Find the points of inflection (if any).

   (e) Sketch the graph of f.

Good Luck